the relativistic ideas of force, work, and energy.

### 4.5.2 Relativistic Force, Work, Kinetic Energy

All these concepts are defined by analogy with their corresponding Newtonian versions. Thus relativistic force is defined as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{4.69}$$

a definition which reduces to the usual Newtonian form at low velocities. This force will do work on a particle, and the *relativistic work* done by $\mathbf{F}$ during a small displacement $d\mathbf{r}$ is, once again defined by analogy as

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{4.70}$$

The rate at which $\mathbf{F}$ does work is then

$$P = \mathbf{F} \cdot \mathbf{u} \tag{4.71}$$

and we can introduce the notion of relativistic kinetic energy by viewing the work done by $\mathbf{F}$ as contributing towards the kinetic energy of the particle i.e.

$$P = \frac{dT}{dt} = \mathbf{F} \cdot \mathbf{u} \tag{4.72}$$

where $T$ is the *relativistic kinetic energy* of the particle. We can write this last equation as

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{u} = \mathbf{u} \cdot \frac{d\mathbf{p}}{dt}$$

$$= \mathbf{u} \cdot \frac{d}{dt} \frac{m_0 \mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

$$= \frac{m_0 \mathbf{u} \cdot \dfrac{d\mathbf{u}}{dt}}{\sqrt{1 - u^2/c^2}} + \frac{m_0 \mathbf{u} \cdot \mathbf{u} u \dfrac{du}{dt}}{c^2 \sqrt{1 - u^2/c^2}}$$

But

$$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = u \frac{du}{dt} \tag{4.73}$$

and hence

$$\frac{dT}{dt} = \left[ \frac{m_0}{\sqrt{1 - u^2/c^2}} + \frac{m_0 u^2/c^2}{\sqrt{(1 - u^2/c^2)^3}} \right] u \frac{du}{dt}$$

$$= \frac{m_0}{\sqrt{(1 - u^2/c^2)^3}} u \frac{du}{dt}$$

so that we end up with

$$\frac{dT}{dt} = \frac{d}{dt} \left[ \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} \right]. \tag{4.74}$$

Integrating with respect to $t$ gives

$$T = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} + \text{constant}. \tag{4.75}$$

By requiring that $T = 0$ for $u = 0$, we find that

$$T = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} - m_0 c^2. \tag{4.76}$$



Interestingly enough, if we suppose that $u << c$, we find that, by the binomial approximation[6]

$$\frac{1}{\sqrt{1 - u^2/c^2}} = (1 - u^2/c^2)^{-\frac{1}{2}} \approx 1 + \frac{u^2}{2c^2} \tag{4.77}$$

so that

$$T \approx m_0 c^2 (1 + u^2/c^2) - m_0 c^2 \approx \tfrac{1}{2} m_0 c^2 \tag{4.78}$$

which, as should be the case, is the classical Newtonian expression for the kinetic energy of a particle of mass moving with a velocity $\mathbf{u}$.

[6] The binomial approximation is $(1 + x)^n \approx 1 + nx$ if $x << 1$.